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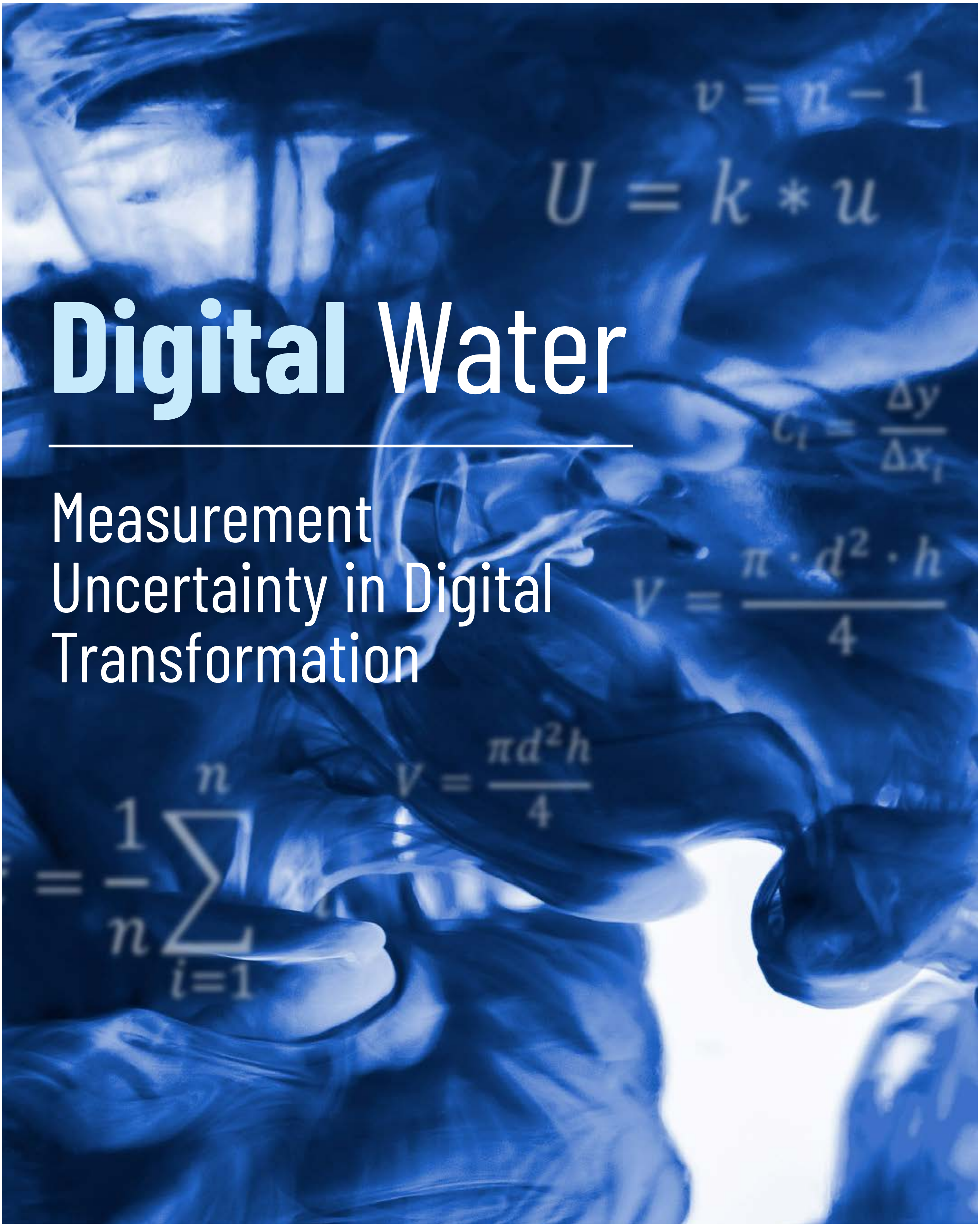
Digital Water

Measurement Uncertainty in Digital Transformation



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Measurement Uncertainty in Digital Transformation

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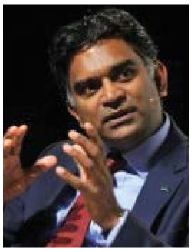
Abbreviations

a	Lower limit of rectangular distribution
b	Upper limit of rectangular distribution
Ci	Sensitivity coefficient of quantity x_i
CV	Coefficient of variation
d	Diameter
h	Height
k	Coverage factor
n	Number of points
RD	Relative deviation
$S(x)$	Standard deviation of measurement x
u	Standard uncertainty
U	Expanded uncertainty
v	Degrees of freedom
V	Volume
x	Quantity being measured
x_i	Result of the i^{th} measurement
Δx_i	Small increment in x_i
y	Output value
Δy_i	The increment in output value caused by Δx_i

Foreword



In an era of rapid technological advancement with digitalisation and digital tools at its core, the water sector is experiencing transformative change.

As we confront the challenges of climate change and population growth, and deal with vital operational issues such as reducing water leakage to optimise use of available resources, the water sector seeks novel solutions. Digitalisation emerges as a key enabler, offering the tools to implement the transformations that enhance efficiency and resilience.

This paradigm shift is about more than merely adopting technology. It represents a strategic effort to bolster the resilience of utilities grappling with multifaceted challenges.

As digital solutions become increasingly integral to daily operations, the understanding and management of uncertainty in measurements emerge as critical elements in ensuring the reliability of instruments underpinning the digitalised sector. This paper explores the crucial role of measurement accuracy in the digital water landscape, addressing the uncertainties inherent in the application of fluid dynamics and providing insights into the various factors influencing uncertainty in flow measurements and how to decrease this uncertainty.

For utilities embarking on their digital journey, it is important to embrace a meticulous assessment of uncertainty analysis focused on the diverse sources of uncertainty embedded in their measurement processes.

This publication provides a contributing voice in the ongoing discourse on digitalisation, being aligned with the International Water Association's Strategic Plan 2019–2024, which emphasises the significance of innovation in addressing global water challenges. The paper emphasises the significance of systematic data collection, and advocates for the adoption of fitting instrumentation.

IWA, through its Digital Water Programme and the Digital Water White Paper Series, is working to guide the water sector through the rapidly changing digital terrain, The Digital Water White Paper Series, including this latest contribution, provides unique perspectives on the intricacies of the evolving digital landscape. Such outputs demonstrate how IWA supports the collective power of shared knowledge and best practices.

By actively engaging the global water community, IWA drives the adoption of an informed approach to water management. Through this collaborative effort, we can strengthen the reliability and resilience of the water sector, thereby ensuring the responsible and sustainable provision of this vital resource to communities worldwide.

Kalanithy Vairavamoorthy
Executive Director of the International Water Association

1. Measurement Uncertainty

It is a popular misconception that measurement is an exact science. In fact, all measurements are merely estimates of the true value being measured which implies some degree of doubt about the accuracy of that measurement itself. For example, the repeated measurement of a fixed quantity will never yield the same result every time. The degree of doubt about the measurement becomes increasingly important with the requirement for increased accuracy. For example, with regards to fluids, the relative cost of the measured fluid would need to be considered: e.g., the measurement of the flow of petroleum has historically been much more accurate than the measurement of water flow for either industrial or domestic supply. **Uncertainty of measurement gives an indication of the quality or reliability of a measurement result.**

Due to issues related to climate change, an increasing population, and the need to reduce leakage levels, **obtaining lower uncertainty flow measurements from water networks has never been more important.** With current plans being made to transfer water between water companies from water rich areas to areas of water scarcity, accurate flow measurement will become increasingly important for custody transfer monitoring of the transferred volumes of water. Improving knowledge of measurement uncertainty, and how this impacts flow measurements, is vital to the efficient operation of the water industry [1].

The purpose of this document is to give the reader an understanding of the factors affecting the accuracy of a measurement, and of the methods used to assess the way in which the various factors contribute to the overall accuracy. This document is by no means a comprehensive review of measurement uncertainty. If more information is required, the reader is asked to consult the ISO/IEC "Guide to the expression of uncertainty in measurement (GUM)".

1.1 Expressing uncertainty

The uncertainty of a measurement is the size of the margin of doubt; in effect, it is an evaluation of the quality of the measurement result produced. To fully express the result of a measurement to reflect its true value, three numbers are required:

(1) **The measured value:** which is simply the figure indicated on the measuring instrument. *(10 m^3/hr in the example below)*

(2) **The uncertainty of the measurement:** i.e., the margin or interval around the indicated value inside which you would expect the true value to lie with a given confidence level. *(± 0.3 m^3/hr in the example below)*

(3) **The level of confidence attached to the uncertainty:** i.e., a measure of the likelihood that the true value of a measurement lies in the defined uncertainty interval. In industry, the confidence level is usually set at 95%. *(95% in the example below)*

For example:

10.0 ± 0.3 m^3/h at 95 % confidence level

1.1.1 Standard and Expanded Uncertainty

The **Standard Uncertainty (u)** defines a narrow band either side of the mean value (or, if appropriate, single value) within which the true value might be expected to lie. Unfortunately, the confidence level attached to this band is low. Assuming a normal distribution, which is explained in Section 2.2.4, we are only 68% confident that the true value will lie within this interval (Figure 1). The **standard uncertainty** is the basic building block of uncertainty used for general uncertainty calculations.

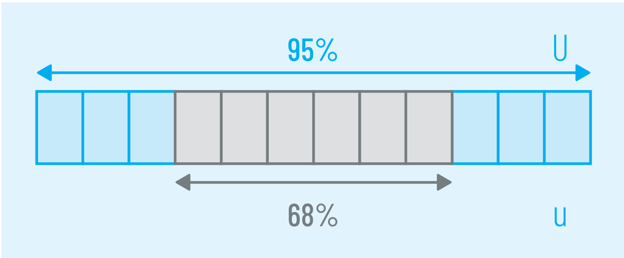


Figure 1 – Uncertainty levels

The use of an uncertainty band with a confidence of only 68% is unacceptably low for most practical measurement situations. A higher confidence level, and therefore a larger uncertainty interval, is required. The larger interval is called the **Expanded Uncertainty (U)**, and normally this is represented as a 95% confidence uncertainty level.

Example 1: Expressing the answer

Suppose we are taking a reading of a flow rate of water in a pipeline. The measured value from the flow meter is 10.0 m^3/h, as Figure 2 shows. It is known for this measurement system that the uncertainty at 95% confidence level is 3%. How do we express this result fully, including the uncertainty?

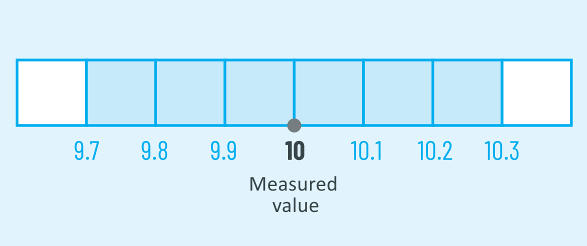


Figure 2 – An illustration of measurement uncertainty

The result of this measurement should be expressed as:

10.0 ± 0.3 m^3/h at 95 % confidence level

Therefore, we know that the:

- Measured value is 10 m^3/h
- Uncertainty of the measurement is 0.3 m^3/h
- Confidence level is 95%

1.1.2 Coverage factor

The multiplier between the Standard Uncertainty and the Expanded Uncertainty at a given confidence level is called the **Coverage Factor (k)**. For the normal distribution, the value of k depends on the confidence level you require, Figure 1.

Thus:

$$U = k * u \quad (1)$$

For a normal distribution the k values used are shown in Table 1:

Table 1: coverage factor (k) for different normal distribution confidence levels

Confidence Level	68 %	90 %	95 %	99 %
Coverage Factor (k)	1.00	1.64	1.96	2.58

1.1.3 What is not considered "uncertainty"

- **Mistakes made by operators** are NOT uncertainties – operator mistakes can be avoided by working carefully through a procedure and checking work.
- **Tolerances** are NOT uncertainties – tolerances are acceptance limits chosen for a process or a product.
- **Specifications** are NOT uncertainties – a specification tells you what to expect from a product.
- **Accuracy** is NOT uncertainty – the true value of a measurement can never be known.

1.2 Error versus uncertainty

Very often people confuse *error* and *uncertainty* by using the terms interchangeably. However, while uncertainty is the margin of doubt associated with a measurement, error is the difference between the measured value and the true value, Figure 3.

Measurements should be **fit for purpose**. For example, if we are fitting curtains in a window our measurement of the window space need not be very accurate. However, if we are fitting a pane of glass in the same window our measurement should be more careful and have a lower value of uncertainty.

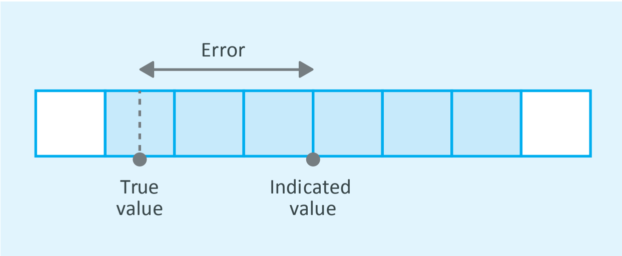


Figure 3 – An illustration of measurement error

1.3 Uncertainty terminology

Accuracy

Often with documentation accompanying an instrument, the accuracy of the instrument is given in numerical terms. This is incorrect, as accuracy is a qualitative rather than a quantitative term. For example, it is perfectly correct to state that one instrument is more accurate than another but wrong to ascribe a number to the accuracy.

Repeatability

This is defined as the closeness of agreement between independent results obtained using the same method on independent test material, under the same conditions (i.e. same operator, same apparatus, same laboratory and after short intervals of time). Accuracy and repeatability are often confused. Results that are accurate are also repeatable but results that are repeatable may not necessarily be accurate, as shown in Figure 4.

We can say that:

- Good accuracy means good repeatability.
- Poor repeatability means poor accuracy.
- Good repeatability does not necessarily mean good accuracy.

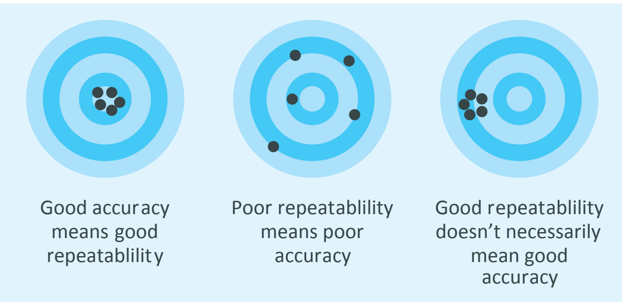


Figure 4 – The relationship between repeatability and accuracy

Reproducibility

Reproducibility is a similar concept to repeatability but applies when the same method is used on identical test material, but in this case under different conditions (e.g., different operator, long time gap between measurements, different test facility etc.). When reproducibility is stated, if the conditions have changed, this should also be clearly stated. Note that if a quantity is reproducible then it is predictable.

Example 2: The effect of errors

In financial terms the expression of uncertainty allows us to estimate the degree of exposure caused by a measurement.

For example, a factory uses 100,000 m^3 of water per year and the water costs £1.50 per m^3. If the flow meter measuring the factory water usage over-reads by 5% the factory is overcharged by £7,500 every year (Figure 5).

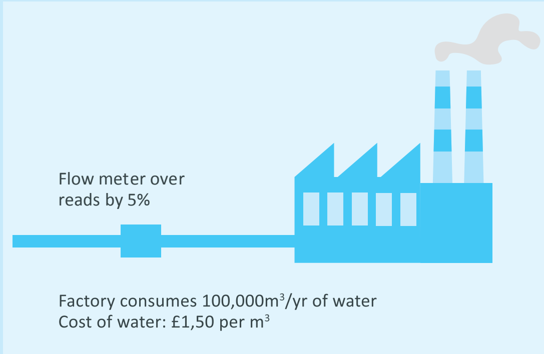


Figure 5 – The effect of errors

Bias errors

A bias error is an error that affects every measurement in the same way. In random error effects, the true value can be at any part of the uncertainty interval. In bias errors, the true value is always in the same position in the uncertainty interval. Bias errors are particularly important in uncertainty analysis, as these combine in such a way as to rapidly build up in magnitude, becoming significant or often becoming the dominant uncertainty source in the system.

As an example of a bias error, consider a domestic water meter. If this is not set such that the meter reads zero when there is no flow within the pipe, then when it is used to measure an actual flow, the measured value will be offset from what it should be. No matter how many times the flow is taken and averaged, the scatter of values would be centred on an offset value. The effect of this offset is inherently present in the result.

1.4 Evaluating uncertainty

The process of evaluating the uncertainty of an individual measurement involves a series of simple and logical steps:

1. Define the relationship between all the inputs of the measurement and the final result. For example, a measurement may have uncertainty in the calibration and the resolution of the measuring instrument.

2. Draw up a list of all the factors that you consider contribute to the uncertainty of the measurement. This may mean that you consult with the operator who is taking the measurement and who best knows the system.

3. For each of the sources of uncertainty that you have identified, make an estimate of the magnitude of the uncertainty.

4. For the relationship described in **STEP 1**, estimate the effect that each input has on the measurement result.

5. Combine all the input uncertainties using the appropriate methodology to obtain the overall uncertainty in the final result.

6. Express the overall uncertainty as an interval about the measured value within which the true value is expected to lie with a given level of confidence.

These steps are summarised in Figure 6:

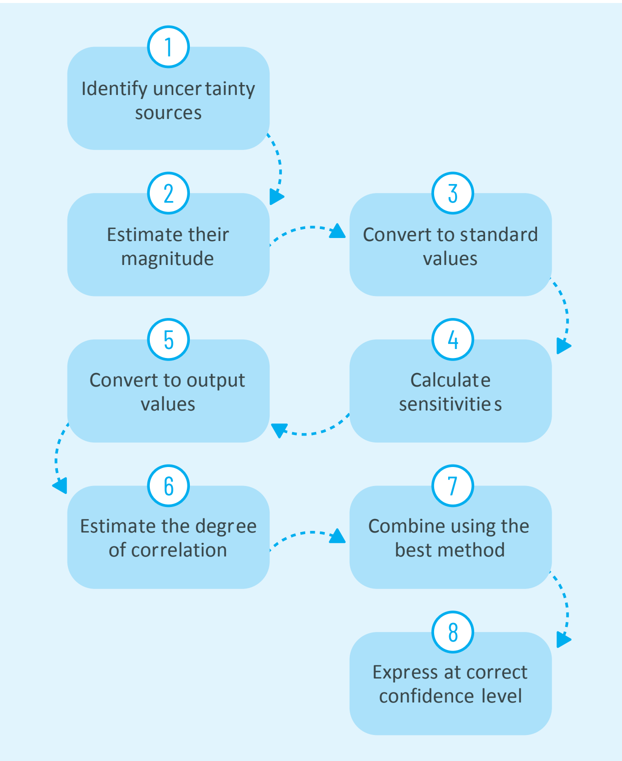


Figure 6 – Summary of standard uncertainty calculation technique

1.5 Common sources of uncertainty

The functional relationship will define all the input variables. The next step is to list all the factors that can influence the measurement of each input. The provision of a definitive list here is impractical and the individual engineer must use his own judgement to decide the factors to consider in any analysis. The essential point is to recognise that the sources of uncertainty extend far beyond the simple aspects of reading the meter output. For example, a flow meter will expand and contract with temperature and, if calibrated at room temperature and then used to meter hot crude oil direct from a wellhead, the fluid temperature will affect the results obtained from the meter.

While it is not possible to list all the likely sources of uncertainty, the following list gives an indication of the types of factors that should be considered.

The Environment – many instruments are sensitive to changes in pressure, temperature, humidity, and vibration. Instruments involving electrical measurement are, in addition, affected by voltage fluctuations and by electromagnetic and radio-frequency interference.

The Measured Quantity – in weighing his tomatoes, the greengrocer has a stable quantity to measure but in many industrial processes the parameter being measured is dynamic. For example, a flow rate may vary as a pump speed varies due to changes in electrical input or as a control valve operates in response to changes in pressure or temperature.

Calibration Uncertainties – when an instrument is calibrated, the uncertainty of the calibration process is transferred to the new instrument and, no matter how carefully that instrument is then used, its uncertainty can never be better than that of the calibration. No calibration certificate should ever be accepted, either from within the company or from an outside calibration laboratory, without a statement of the uncertainty of the calibration.

Operator Bias – although industrial measurement is increasingly automated, it is surprising how many readings are still taken manually. Wherever this happens, human judgement is used, and the results will be subject to additional uncertainty. For instance, parallax will affect the reading differently, depending on whether the meter is above or below the operator's eye-line or whether it is to his left or right. Fluctuating readings are particularly susceptible to operator differences as everyone has their own technique for reading a flickering dial needle.

The Instrument Used – many aspects of the instrument being used affect the result of the measurement: resolution, bias, hysteresis, ageing, non-linearity etc.

The Measurement Procedure – measurement procedures should be fully documented in company quality procedures and steps taken to ensure that the procedures are adhered to properly. However, there will be occasions on which different instruments must be used or readings taken over a different time interval. The impact of this on the uncertainty should be assessed when applicable.

Usage Effects – it is common practice to calibrate an instrument in ideal conditions rather than in the conditions of its normal usage. Thus, a flow meter may be calibrated in a well-conditioned flow, remote from disturbances such as valves or bends. However, such disturbances can distort the flow pattern and so corrupt the reading of an ultrasonic meter that assumes an ideal flow pattern, or they may introduce swirl that increases or decreases the rotational speed of a turbine meter. In some cases, the meter itself may disturb the measurement, when, for instance, an insertion flow meter causes a blockage and increases the local velocity. In some cases, it may be possible to calibrate in situ or in a similar installation, but very often this will not be practical and the impact of these effects on the measurement must be assessed.

Data Collection and Processing – parameters such as density or chemical composition are often measured by withdrawing a small sample from the flow or batch and assuming that its density or composition is a good representation of that of the whole flow. Poor mixing can undermine this assumption and the effect of this will need to be quantified. When a time-varying quantity is being measured at intervals, it is important that the sampling interval is chosen to obtain a good average and care is needed to avoid the type of bias errors that arise when a sinusoidal signal is sampled at the same frequency as its fluctuations, as this can result in the measured values representing only the peaks or troughs. In these circumstances the sampling rate should typically be at least four times the signal frequency. Data collection uncertainties can also result from the resolution of computer logging hardware (e.g. analogue-to-digital converters). Processing uncertainties can result from round-off errors within software or from values of physical properties taken from reference books and embedded within the calculation process.

This list is intended only as a guide to the types of sources that the reader should be aware of in drawing up the source list for each input. Every measurement engineer should be aware of the possibility of additional sources that may be peculiar to the process being analysed.

1.6 The cost of uncertainty

In all measurement systems, it is generally the case that the lower the required measurement uncertainty, the higher the financial cost required to achieve it, as shown in Figure 7. Thorough evaluation of the system uncertainty helps ensure that the metering system is properly designed, cost effective and fit for purpose, meeting any uncertainty constraints specified legally, through partner agreements or through internal requirements.

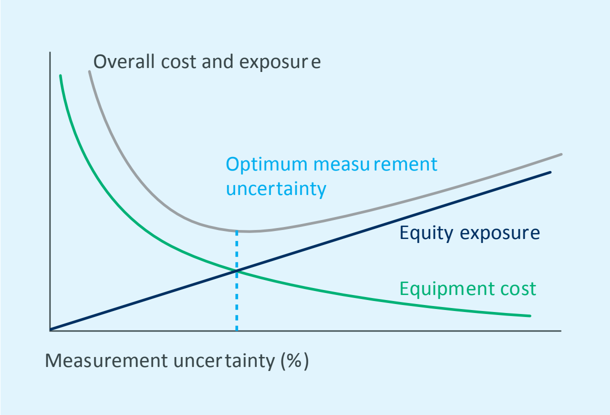


Figure 7 – Uncertainty versus cost

Example 3: The cost of uncertainty

For example, suppose a farmer is filling a water tanker with water, the uncertainty in the flow measurement of water is 2% at the 95% confidence level. If the capacity of the water tank is 2000 m^3, what is the uncertainty of half filling the tank? The volume of flow is given by:

1000 ±20 m^3 at 95 % confidence level

If the cost of the water is £1.50 m^3, the cost of half filling the water tank is:

£1500 ± £30 at 95 % confidence

2. Calculation Methods

2.1 Introduction

The ISO/IEC Guide [1] specifies two distinct methods of uncertainty analysis Type A and Type B analyses, (Figure 8). Type A is based upon the statistical analysis of multiple readings of the same measurement, whereas Type B is essentially a non-statistical approach. In most analyses we usually have to apply a mixture of both types to arrive at a solution.

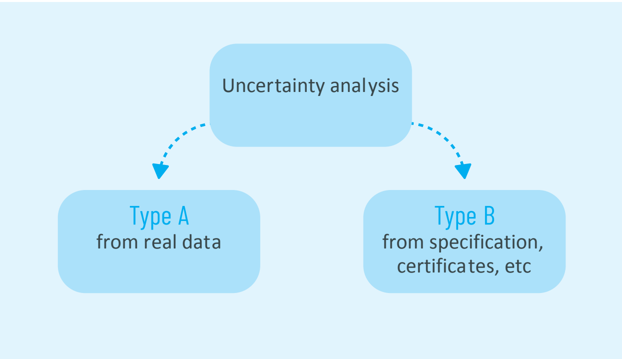


Figure 8 – Uncertainty analysis

2.2 Type A analysis

2.2.1 Arithmetic mean

When you take repeated measurements of a nominally constant quantity you will never get exactly the same results. Due to the random fluctuations inherent in any measurement, there will always be some differences in the results. If you are taking repeated measurements, then the best estimate of the true value is the average, or the arithmetic mean $\bar{x}$ of a quantity x.

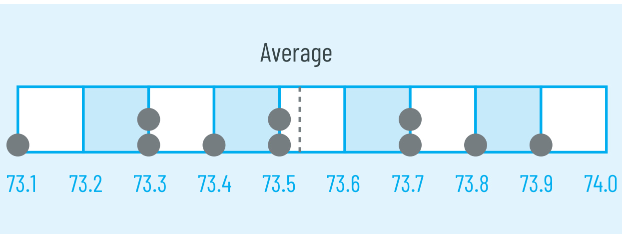


Figure 9 – A set of measurements illustrating the average value

The average is simply calculated by adding up all of the results in the test series and dividing by the number of points taken:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (2)$$

Where n is the number of points. One of the most commonly asked questions, when carrying out this type of experiment is, “How many points should I take to get a good value?” The more points you take, the more confidence you will have that the mean is closer to the real value. However, acquiring a lot of points takes time and money. It is normally better to compromise between taking too many and too few. A good target is to take about 10 measurements.

2.2.2 Spread or standard deviation

As well as the average value of a set of measurements, it is also useful to know what the spread of the measurements are. This gives an indication of the uncertainty of the measurement. One measure of spread is the ‘range’. The ‘range’ is the value of the largest minus the value of the smallest measurement. This has the limitation that the ‘range’ misses out the majority of the data and so does not account for the scatter of the set. The most commonly used method of measuring spread is to calculate the ‘standard deviation’, which is based on the number of points taken. The formula for the standard deviation $s(x)$ of a measurement x is given by:

$$s(x) = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} \quad (3)$$

Where n is the total number of measurements taken, x_i is the result of the i^{th} measurement and $\bar{x}$ is the arithmetic mean of the n measurements. Note that in this formula we divide by $n-1$ rather than n. This is because we are calculating an estimate of the standard deviation based on a sample of n rather than the entire population of readings, as the entire population of readings can be a very high number of flow measurements.

The standard deviation is often expressed as a proportion of the mean. This is called the coefficient of variance (CV) and is defined as:

$$CV = \frac{s\,(x)}{\bar{x}} \quad (4)$$

It can also be expressed as the percentage deviation or relative deviation (RD):

$$RD = 100\frac{s(x)}{\bar{x}} \quad (5)$$

Care is needed when using relative terms as some measurements, such as temperatures, have arbitrary zero points and these lead to nonsensical relative deviations.

2.2.3 Degrees of freedom

As described earlier, the standard deviation provides a measure of the scatter of the data and, as with any measurement, the more data there is, the more reliable the measurement is. The reliability of the standard deviation as a measure of the scatter is defined by a statistic called the *degrees of freedom*, which is defined as:

$$v = n - 1 \quad (6)$$

The standard uncertainty defines a narrow band either side of the mean value (or, if appropriate, single value) within which the true value might be expected to lie. Unfortunately, the confidence level attached to this band is low. The level depends on the number of measurements involved in deriving the standard deviation, or more precisely on the degrees of freedom of the standard deviation.

To increase the chances of the true value lying within the quoted band, the bandwidth is often extended by multiplying the standard uncertainty by a coverage factor, k, whose value reflects confidence in the standard uncertainty, as defined by the degrees of freedom. The confidence required in the expanded uncertainty is thus:

$$U_x = k * u_x \quad (7)$$

The value of k is taken for the appropriate degrees of freedom and required confidence level from Table 2. The coverage factor is also known as “Student’s t” from the original publication of the table by W. Gosset, a statistician writing under the pseudonym of “student” [2].

2.2.4 Normal or Gaussian distribution

Very often when a measurement is being made, most of the readings will fall close to the average value with a few falling further away. Assuming a significantly large set of measurements will give rise to the characteristic bell-shaped curve as shown in Figure 10.

Here z (on the x axis) is a number which indicates how many standard deviations above or below the mean value that a particular reading or value is. An example of this type of distribution would be the spread of heights of men in the UK. Most will have heights near the average, but a few will be considerably taller or shorter. In this type of distribution, 68% of men’s heights will be within a single standard deviation of the mean, while 95% will lie within two standard deviations.

Figure 10 also shows the confidence levels and coverage factors associated with the standard deviations of a normal distribution.

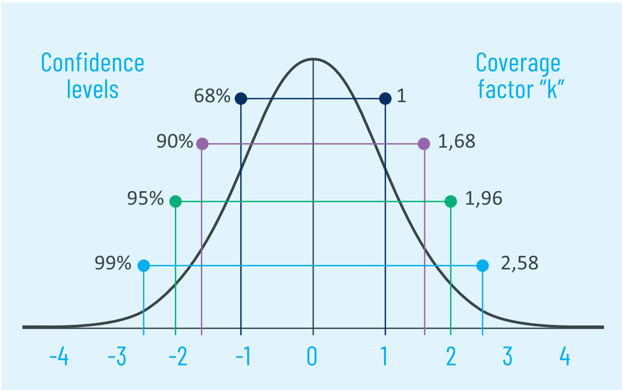


Figure 10 – Standard normal or gaussian distribution curve

Table 2 – Table of coverage factor (k) or student's t values

Degrees of Freedom	Confidence Level		
	90 %	95 %	99 %
	k	k	k
1	6.31	12.71	63.66
2	2.92	4.30	9.92
3	2.35	3.18	5.84
4	2.13	2.78	4.60
5	2.02	2.57	4.03
6	1.94	2.45	3.71
7	1.89	2.36	3.50
8	1.86	2.31	3.36
9	1.83	2.26	3.25
10	1.81	2.23	3.17
11	1.80	2.20	3.11
12	1.78	2.18	3.05
13	1.77	2.16	3.01
14	1.76	2.14	2.98
15	1.75	2.13	2.95
16	1.75	2.12	2.92
17	1.74	2.11	2.90
18	1.73	2.10	2.88
19	1.73	2.09	2.86
20	1.72	2.09	2.85
25	1.71	2.06	2.79
30	1.70	2.04	2.75
40	1.68	2.02	2.70
60	1.67	2.00	2.66
100	1.66	1.98	2.63
∞	1.64	1.96	2.58
Degrees of Freedom	**Confidence Level**		
	68.27 %	**95.45 %**	**99.73 %**
	k	**k**	**k**
∞	1.00	2.00	3.00

2.3 Type B analysis

Often it is impossible to assess the magnitude of the uncertainty from repeated measurements and it has to be quantified using other means. For example, these could be:

- The uncertainty quoted on a calibration certificate.
- Engineering judgement based on experience of a measurement system.
- Manufacturer's specifications.

In making Type B assessments, it is necessary that all of the measurements should be at the same confidence level, so that the uncertainties can be compared and combined. This will usually be the *standard uncertainty* which is equivalent to the standard deviation for a normal distribution. Type B assessment is not necessarily governed by the normal distribution, and the uncertainties may be quoted at a range of confidence levels. Thus, a calibration certificate may give the meter factor for a turbine meter with 95% confidence, while an instrument resolution uncertainty defines, with 100% confidence, the range of values that the measurement could be. These higher confidence uncertainties are known as *expanded uncertainties U(x)* and are related to the standard uncertainty *u(x)* by the expression.

$$U(x) = k * u(x) \quad (8)$$

where k is known as the coverage factor, which is a multiplier to reflect the degree of confidence of the possible range of results. The most common example of a Type B assessment with a normal distribution would be a calibration certificate quoting a percentage confidence level or a *k* factor.

Example 4: Mean, variance, standard deviation, degrees of freedom, coefficient of variation and relative deviation

Toluene is being used as a feedstock in a petrochemical plant and the flowrate is measured using a turbine meter. To reduce Type A uncertainties in the flowrate measurements each reading used for control purposes is derived from five individual readings. A typical set of values is given in Table 3. Calculate the mean, standard deviation, degrees of freedom, coefficient of variation and relative deviation:

Reading No.	1	2	3	4	5
Flow (l/s)	122.7	123.2	122.3	122.8	**123.0**

Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{(122.7 + 123.2 + 122.3 + 122.8 + 123.0)}{5} = 122.8\ l/s \quad (9)$$

Variance

$$S_s^2 = \frac{1}{n-1} * \sum_{i=1}^{n}(x_i - \bar{x}) = \frac{1}{5-1} * ((122.7 - 122.8)^2 + \cdots + (123.0 - 122.8)^2) = 0.115 \quad (10)$$

Standard deviation

$$S_s = \sqrt{S_s^2} = \sqrt{0.115} = 0.339 \quad (11)$$

Degrees of freedom

$$v = n - 1 = 5 - 1 = 4 \quad (12)$$

Coefficient of variation

$$CV = \frac{S_s}{\bar{x}} = \frac{0.339}{122.8} = 0.00276 \quad (13)$$

Relative deviation

$$RD = 100\frac{S_s}{\bar{x}} = 100\ x\ \frac{0.339}{122.8} = 0.276\ \% \quad (14)$$

2.3.1 Rectangular distribution

A rectangular distribution (Figure 11) is one for which the probability of occurrence is the same for all values of a measurement. It is sometimes called a *uniform distribution.* For example, if a fair die is thrown, the probability of obtaining any one of the six possible outcomes is 1/6. Since all of the outcomes are equally probable, the distribution is rectangular.

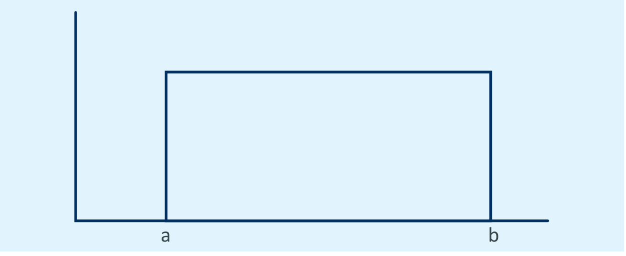


Figure 11 – Rectangular or uniform distribution

A common example of this type of distribution is the uncertainty caused by the resolution of an instrument. If a meter reads the flow rate as 3.5 m^3/h to a single decimal place, then the true value could lie anywhere between 3.45 and 3.55 m^3/h with equal probability. To convert the range to the standard uncertainty required for comparison and calculation the following formula is used.

$$u(x) = \frac{b-a}{\sqrt{12}} \quad (15)$$

2.3.2 Skewed distributions

These are non-symmetrical distributions where one tail is longer than the other.

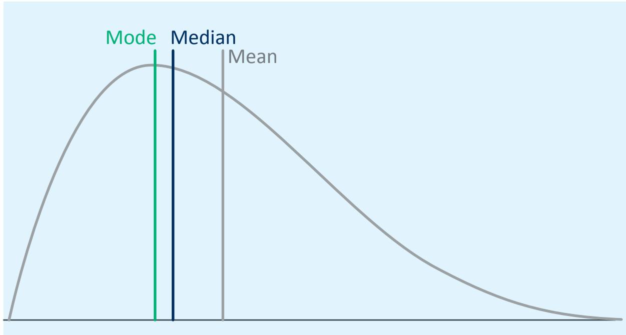


Figure 12 – A positively skewed distribution

A *positively skewed* distribution means that the tail is long at the upper end of the range (Figure 12). They are much more common than negatively skewed distributions. An example of a positively skewed distribution is the spread of salaries in a typical company. Most employees will be paid a salary that lies reasonably close to the mode (the most popular salary band). Note that the median is defined as the measurement for which there is an equal number of measurements of greater and smaller value. However, a few employees at senior levels in the company will be paid considerably more than those in the modal band. Although the amount of these people will be small (hence the tail) they will have the effect of increasing the mean salary. This makes the distribution positively skewed. A negatively skewed distribution is like a mirror image of the positively skewed, the tail is at the lower end of the value range. An example of this is a set of scores in an easy test, where most people score high, but some less able pupils get a low score. Therefore, the tail is at the lower end of the scoring range.

3. Combining uncertainties

Before you are able to combine uncertainties from the various sources to get an overall uncertainty for a given quantity, a series of criteria need to be met and calculations performed.

For example, the volume of a water storage tank is given by:

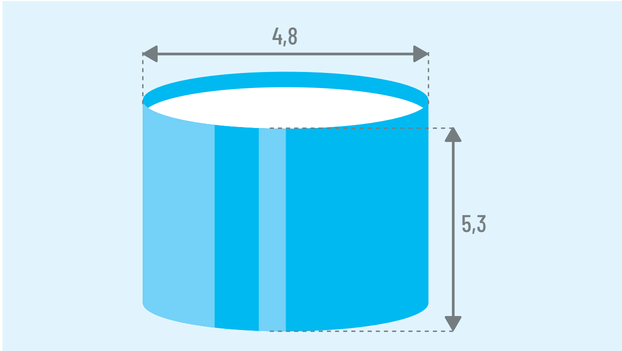


Figure 13 – Volume of a storage tank

$$V = \frac{\pi d^2 h}{4} \quad (16)$$

Where *h* is the height of the tank and *d* is the diameter. Each of these quantities should be measured before the tank volume can be calculated. At this stage, the magnitude of the uncertainty of each of the measurable quantities should be evaluated. This may be done by using the manufacturers specification on the measuring instrument or by using engineering judgement (Type B). The uncertainty may also be evaluated by making repeated measurements of the quantity and conducting a statistical analysis of the results (Type A).

Most uncertainty analyses consist of a mixture of Type A and B. On many manufacturers specifications or calibration certificates, the uncertainty of a measurement device is expressed as an expanded uncertainty (U) at a 95% confidence level. In Type A evaluations the standard deviation is usually calculated. Before combination, each uncertainty should be reduced to at least a common confidence level and at best to a standard uncertainty (equivalent to a standard deviation or u). To add uncertainties together they must be in the same units. The most appropriate units are those of the output quantity whose uncertainty we wish to calculate.

3.1 Sensitivity coefficients

Conversion to output units is done by calculating sensitivity coefficients for each source with respect to the output quantity. Analytically this is done by partial differentiation of the output with respect to the specified quantity. Alternatively, in situations where the governing equation is complicated, this may be accomplished numerically by adding and subtracting increments from the specified quantity to gauge the effect on the output.

3.1.1 Analytical method

When the functional relationship is specified, the sensitivity coefficient is defined as the rate of change of an output quantity y with respect to an input quantity x_i and the value is obtained by partial differentiation:

$$C_i = \frac{\partial y}{\partial x_i} \quad (17)$$

However, when non-dimensional uncertainties (for example percentage uncertainties) are used, non-dimensional sensitivity coefficients must also be used:

$$C_i^* = \frac{\partial y}{\partial x_i} . \frac{x_i}{y} \quad (18)$$

Example 5: Sensitivity coefficients by the analytical method

The analytical method is by differentiation of the relevant formula. For example, suppose we wish to calculate the uncertainty in the volume of a cylindrical water storage tank of diameter 4.8 m and height 5.3 m. This uses the equation.

$$V = \frac{\pi \cdot d^2 \cdot h}{4} \quad (19)$$

The uncertainty in the volume will be caused by uncertainties in the measurements of the diameter and the height. The dependence of the uncertainty in the volume on the uncertainty of the diameter is given by;

$$C_d = \frac{\partial V}{\partial d} = \frac{2\pi \cdot d \cdot h}{4} = \frac{1}{2} . \pi dh = \ 39.96 \ \frac{m^3}{m} \quad (20)$$

and the uncertainty of the volume on the height:

$$C_h = \frac{\partial V}{\partial h} = \pi \frac{d^2}{4} = 18.10 \ \frac{m^3}{m} \quad (21)$$

3.1.2 Numerical method

When no mathematical relationship is available, or the functional relationship is excessively complicated, it may be easier to obtain the sensitivity coefficients numerically, by calculating the effect on a small change in the input variable x_i on the output value y. This calculation can be broken down into a number of stages:

1. Calculate y using x_i
2. Recalculate using:

$$x_i + \frac{\Delta x_i}{2} \quad (22)$$

Where Δx_i is a small increment in x_i.
The result of the re-calculation can be expressed as:

$$y + \Delta y_i \quad (23)$$

Where Δx_i is the increment in y caused by the addition of Δx_i

3. Recalculate using:

$$x_i - \frac{\Delta x_i}{2} \quad (24)$$

Where Δx_i is a small increment in x_i.
The result of the re-calculation can be expressed as:

$$y - \Delta y_i \quad (25)$$

Where Δy_i is the increment in y caused by the addition of Δx_i.

4. Calculate the difference between the upper and lower y from:

$$\Delta y = \Delta y_1 + \Delta y_2 \quad (26)$$

5. The sensitivity coefficient can be calculated from:

$$C_i = \frac{\Delta y}{\Delta x_i} \quad (27)$$

The increment used should be as small as practical, considering the possibility that instability may occur if the increment results in changes in the calculated result. Often it is chosen to be the standard uncertainty of the measurement. The problem can generally be avoided by checking the stability of the sensitivity coefficient over a range of increments. The value of the increment chosen should typically be no larger than the uncertainty in the parameter x_i.

Example 6: Sensitivity coefficients by the numerical method

The water tank mentioned above has a diameter of 4.8 m and a height of 5.3 m. The volume is therefore given by:

$$V = \frac{\pi \cdot d^2 \cdot h}{4} \quad (19)$$

The numerical method of determining the sensitivity coefficient is:

- Add and subtract an increment to the value of the diameter:

$$\frac{\Delta d}{2} = 0.01 \quad (28)$$

The value should be equal to the standard uncertainty of the measurement.

- Calculate the volume with both the larger and smaller values:

$$V_{+} = \pi . \frac{\left(d + \frac{\Delta d}{2}\right)^2}{4} . h = 96.31\, m^3 \quad (29)$$

$$V_{-} = \pi . \frac{\left(d - \frac{\Delta d}{2}\right)^2}{4} . h = 95.51\, m^3 \quad (30)$$

- Calculate the difference in these two volumes:

$$\Delta V = V_{+} - V_{-} = 96.31 - 95.51 = 0.8\, m^3 \quad (31)$$

- Calculate the sensitivity coefficient for the diameter:

$$C_d = \frac{\Delta V}{\Delta d} = \frac{0.8}{0.02} = 40\, m^2 \quad (32)$$

The above procedure then needs to be repeated for the height:

- Add and subtract an increment to the value of the height:

$$\frac{\Delta h}{2} = 0.01 \quad (33)$$

The value should be equal to the standard uncertainty of the measurement.

- Calculate the volume with both the larger and smaller values:

$$V_{+} = \pi . \frac{d^2}{4} . \left(h + \frac{\Delta h}{2}\right) = 96.09\, m^3 \quad (34)$$

$$V_{-} = \pi . \frac{d^2}{4} . \left(h + \frac{\Delta h}{2}\right) = 95.73\, m^3 \quad (35)$$

- Calculate the difference in these two volumes:

$$\Delta V = V_{+} - V_{-} = 96..09 - 95.73 = 0.36\, m^3 \quad (36)$$

- Calculate the sensitivity coefficient for the diameter:

$$C_d = \frac{\Delta V}{\Delta h} = \frac{0.36}{0.02} = 18\, m^2 \quad (37)$$

These agree closely with the analytical values shown in Example 5.

3.2 Contributions to output uncertainty

After the sensitivity coefficients have been calculated it is then possible to evaluate the contributions of each of the uncertainty sources to the overall output uncertainty. The quoted uncertainty should be reduced to a standard value and then multiplied by the appropriate sensitivity coefficient.

$$u_{output} = C_i \,.\, u_i \quad (38)$$

Where u_i is the i^{th} standard input uncertainty source. The units of the output uncertainty are that of the output value.

3.3 Root sum squared method

After all the individual uncertainties have been reduced to a single confidence level, the next stage is to combine them to get an overall uncertainty for the quantity that we are interested in. This is normally completed using the **root sum squared (RSS)** method or in combination with quadrature. When we define an uncertainty interval for a measurement, its true value can theoretically lie anywhere within that interval, with a probability defined by the distribution. So, for a normal distribution, the true value is more likely to lie in the middle of the distribution near the mean value, whereas a rectangular distribution can lie anywhere within the interval with equal probability. For two separate, uncorrelated sources of uncertainty, the true value can lie in any part of their uncertainty intervals and so simply adding them together would not give a representative value of the combined uncertainty.

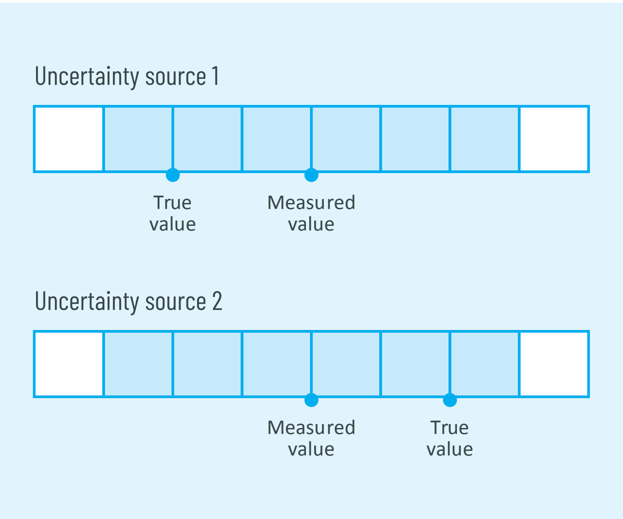


Figure 14 – Combining uncertainties by RSS

For each separate uncertainty source, the true value can either exceed or be smaller than the measured value. The uncertainty for each source should be squared and then added together for all sources. Finally, the square root of this sum should be taken. It is important to be aware that this summation process can only take place when the uncertainties are:

(1) expressed in terms of the derived quantity
(2) all at the same confidence level:

$$u(y) = \sqrt{\sum_{j=1}^{r} \left(C_i \cdot u_j\right)^2} \quad (39)$$

where *y* is the derived quantity and each of the standard uncertainties are denoted by u_j.

3.4 Correlated uncertainties

One or more uncertainties from a particular source are said to be correlated if one depends in some way on the other. For example, if two flow meters are calibrated together against the same standard, if one has a bias error of 0.1% due to the calibration, then so will the other. Equally if the same instrument is used to make multiple measurements, the calibration error in these measurements will be the same. However, for both of these measurements, other sources of uncertainty will be uncorrelated (such as the resolution uncertainty) and partially correlated (such as environmental effects) so that one cannot claim that the overall uncertainty is correlated.

Since for correlated sources of uncertainty, the errors are related in some way, correlated uncertainty sources are not combined in the same way as uncorrelated sources. For instance, if the calibration bias error is the same for two measurements, then the true value of the measurement will be in the same part of the uncertainty interval and will be combined by straight addition. So, for positive correlation, where the sensitivity coefficients have the same sign, the uncertainty is greater than uncorrelated. For negative correlation, where the sensitivity coefficients have opposite signs, the uncertainty will be less than that for uncorrelated sources.

If you are not sure if the sources are correlated, if the correlation would be positive, assume correlated. If the correlation would be negative, assume uncorrelated.

Table 4 – Example of uncertainty calculation

Measurement	Value	Unit	Source of uncertainty	*U*	*k*	*u*	*c*	*c*u*	*(c*u)²*
Diameter	4.8	m	Calibration						
		m	Repeated meas.						
		m	Drift (instrument)						
Height	5.3	m	Calibration						
		m	Resolution						
		m	Drift (instrument)						
Volume	95.91	m^3							

Alternatively, one can undertake two analyses; one where we assume the sources are correlated and another where we assume uncorrelated. If the answers are close, select the larger of the two. If there is a significant difference in the outcome, consult the "Guide to expression of uncertainty in measurement" (GUM) [1].

4. The standard (GUM) method of uncertainty

For a more detailed explanation of all the steps and calculations required to undertake an uncertainty calculation, the reader is asked to consult the ISO/IEC "Guide to the expression of uncertainty in measurement" (GUM). The following example shows a simple case of how the uncertainty of a measurement can be developed.

To undertake a measurement uncertainty calculation a few rules must be followed:

- Can only add like to like
  - Keep to absolute (or relative) uncertainties
- Can only add uncertainties at a common confidence level
  - Therefore we must reduce all uncertainties to standard uncertainties
- Can only add in common units
  - Cannot add kg and °C
- Express everything in terms of contribution to final uncertainty (i.e. m^3 in our example)

Table 4 shows how the uncertainty budget calculation should be determined. The individual sources of uncertainty are calculated horizontally along the rows of the table. This allows all sources of uncertainty to be at a standard level of uncertainty (u). The overall uncertainty is then calculated by adding the values of $(c.u)^2$ vertically down the table and then calulating each of the values listed in the table by following the arrows shown in Table 4. This allows the overall uncertainty of the measurement to be calculated.

Where:
U – Expanded uncertainty (usually quoted at 95% confidence limits)
k – Coverage factor (for 95% confidence levels this equals 1.96)
u – Standard uncertainty (68% confidence level)
c – Sensitivity coefficient

Where the sensitivity coefficient represents how the variables in an equation or function are related to the calculated result, when a small change is made in the variable x in an equation it will have an effect on the magnitude of the result y. So, in the example, by making small changes in both the height and diameter, it is possible to determine the sensitivity of the variable (height or diameter) to the overall result (volume).

When required to calculate the uncertainty in the **volume** of a cylindrical water tank, the height of the tank is nominally 5.3 m and its diameter is 4.8 m.

$$V = \frac{\pi d^2 h}{4} \quad (40)$$

To do this we must measure both the height and diameter of the vessel. The sources of uncertainty are therefore those in the measurement of height and diameter.

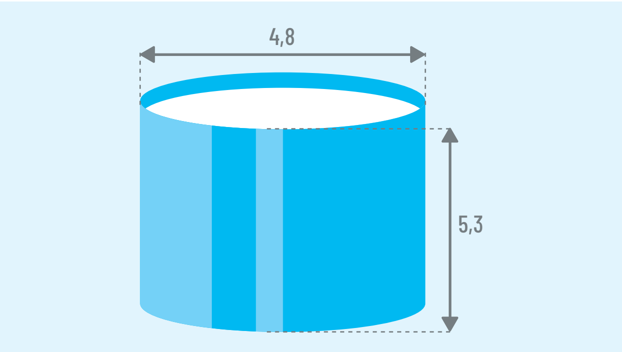


Figure 15 – Volume of water tank

4.1 Height measurement

The principal sources of uncertainty in this measurement are the calibration of the measuring instrument, its resolution, and any changes in the height due to time drift.

4.2 Diameter measurement

The uncertainty sources in the diameter are similar to those in the height measurement. The uncertainty in the instrument calibration and temporal variation in the diameter are both important. The tank may not always be a perfect cylinder, so the diameter may vary depending on which part of the circumference the measurement is made. The diameter may also vary depending on the height at which the reading is taken. As normal, the resolution of the measuring instrument is also an important source. In this case, for convenience, the final three sources are combined into a single source and the standard uncertainty is determined directly.

4.3 Tank volume uncertainties

Summarising the information contained above, the principal tank volume uncertainty sources are shown in Table 5.

Table 5 – Tank volume uncertainty sources

Measurement	Source
Diameter	Calibration
	Determination
	Time drift
Height	Calibration
	Determination
	Time drift

Table 6 – Overall tank volume uncertainty budget

Measurement	Value	Unit	Source of uncertainty	*U*	Probability distribution	*k*	*u*	*c*	*c*u*	*(c*u)²*
Diameter	4.8	m	Calibration	0.048	Normal	1.96	0.024	39.96	0.98	0.958
		m	Repeated meas.	0.021	Normal	1	0.021	39.96	0.84	0.704
		m	Drift (instrument)	0.005	Rectangular	1.73	0.003	39.96	0.12	0.013
Height	5.3	m	Calibration	0.1	Normal	1.96	0.051	18.10	0.92	0.852
		m	Resolution	0.025	Rectangular	1.73	0.014	18.10	0.26	0.068
		m	Drift (instrument)	0.01	Rectangular	1.73	0.006	18.10	0.10	0.011
Volume	95.907	m^3		**3.16**	Normal	1.96	1.62	1	1.62	2.61

4.5 Overall tank volume uncertainty budget

Table 6 lists all the sources of uncertainty in the diameter, height, and volume variation measurements and gives you the uncertainty of the volume measurement of the storage tank, in this case the uncertainty is:

95.91 ± 3.16 m^3 at 95 % confidence

4.6 The importance of uncertainty in measurement

Uncertainty analysis is an essential component of the design and use of any measurement system. Without a thorough uncertainty analysis, time and money will be wasted on inappropriate instrumentation. As demonstrated in the previous sections, the techniques used for performing the analysis are not complicated but must be based on the solid foundation of a detailed review of the whole measurement process.

5. References

[1] E. Belia, B. M. Neumann, L. Benedetti, B. Johnson, S. Murthy, S. Weijers and P. A. Vanrollegham, “Uncertainty in wastewater treatment, design and operation,” IWA Publishing, 2021.

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